tion, cm./sec.

v_r = molar average velocity perpendicular to the cylinder wall, cm./sec.

v_z = molar average velocity parallel to the cylinder wall, cm./sec.

w = mass flow rate, g./sec.

y = dimensionless longitudinal coordinate, z/r_w

z = length dimension, cm.

Δy = distance between consecutive points in the y direction in the finite difference grid

Δs = distance between consecutive points in the s direction in the finite difference grid

μ = viscosity, g.cm./sec.

ρ = density, g./cc.

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Methods for Solving the Boundary Layer Equations for Moving Continuous Flat Surfaces with Suction and Injection

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Several methods that can be used to obtain solutions to the laminar boundary layer momentum, energy, and diffusion differential equations for moving continuous flat surfaces with suction and injection are presented. Results are obtained for a wide range of the injection parameter, $f(0)$, at Prandtl and Schmidt numbers of 1, 10, and 100. Those methods which permit hand calculation of the properties of interest are compared using the numerical solutions of the boundary layer differential equations as the exact solutions. The new integral method of Hanson and Richardson which gives results for the momentum thickness that deviate less than 2.2% from the exact values is recommended for predicting values of the momentum boundary layer parameters. The Von Karman-Pohlhausen method, which was modified to account for suction and injection, is most generally valid. This method gives acceptable values of the transfer coefficients for heat, mass and momentum transfer for most of the values considered.

Several new and modified methods for solving differential equations describing the laminar boundary layer behavior on a moving continuous flat sheet with suction and injection are presented. Emphasis is placed on those methods which permit hand calculation of the properties of interest.

The boundary layer adjacent to continuous moving solid surfaces was first investigated by Sakiadis (*1*). In his pioneering paper the boundary layer differential equations for continuous moving flat and continuous moving cylindrical surfaces were presented. The differential equations describing boundary layer behavior on these continuous surfaces were shown to be the same as the equations describing the behavior of a boundary layer next to a geometrically similar object of finite length. However, the boundary conditions for these two different classes of problems are different. Sakiadis' paper also contained the development of the integral momentum equation for both flat and cylindrical continuous surfaces.

In a subsequent publication Sakiadis (*1*) presented an exact and an approximate solution for the case of laminar flow and flat geometry. An approximate solution thought to be valid for turbulent flow and flat geometry was also given. Later, Sakiadis (*1*) treated the case of a laminar and a turbulent boundary layer on a continuous circular cylinder by an integral method.

Koldenhof (*2*) working independently used an integral technique somewhat different from that used by Sakiadis to treat the problem of a laminar boundary layer on a continuous moving cylinder. The equations of Koldenhof are essentially the same as Sakiadis. Koldenhof also presented the results of an experimental investigation of drag on continuous cylinders and good agreement was obtained between the theoretical and the experimental results.

Griffith (*3*) presented experimental data for shear stress and velocity profiles in the boundary layer adjacent to continuous moving cylindrical surfaces. These data qualitatively agreed with the data of Koldenhof. Griffith also reasoned that for the boundary layer associated with polymer spinning, the boundary layer equation of energy could

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be linearized with little loss of accuracy. Griffith presented solutions to the linearized equation of energy in his paper.

Erickson, Fan, and Fox (4) treated the problem of a laminar boundary layer on a continuous flat surface with suction and injection. This paper theoretically investigates the energy and concentration boundary layer.

Erickson, Cha, and Fan (5) investigated the cooling of a moving continuous flat sheet as it passes through a tank of fluid. Both a finite difference and an integral solution are presented. Equations are developed for predicting the local rate of heat transfer and the temperature of the sheet.

Tsou (6) theoretically and experimentally investigated the velocity field, hydrodynamic stability, and heat transfer for boundary layer flow along a continuous flat surface. He considered both laminar and turbulent boundary layers.

Since obtaining a solution for the boundary layer momentum equation involves solving a two point boundary value problem over an indefinite range of the independent variable, the numerical solution is time consuming at best. Therefore, it is appropriate to develop and examine techniques that will yield results with considerably reduced labor. Such methods are used to obtain solutions to the momentum equation for a range of the injection parameter, f_w, from -1 to 1. These results are then used with the energy and diffusion equations to predict heat and mass transfer coefficients for the Prandtl and Schmidt numbers of 1, 10, and 100 for the same range of the injection parameter.

THE BOUNDARY LAYER EQUATIONS

The heat, mass, and momentum transfer associated with a continuous flat surface that is moving with a constant velocity through a fluid medium at rest is considered. It is assumed that steady, two dimensional, incompressible flow, with a nonzero transverse velocity component at the solid surface occurs, as shown in Figure 1. Constant fluid properties and negligible viscous dissipation are assumed. A stationary frame of reference with its center point located at the slot is used. The positive x axis is parallel to the moving surface and in the same direction. The positive y axis extends upward from the upper surface ($y = 0$). Under these conditions the boundary layer equations of continuity, momentum, energy, and diffusion become:

Continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (1)$$

Momentum

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \quad (2)$$

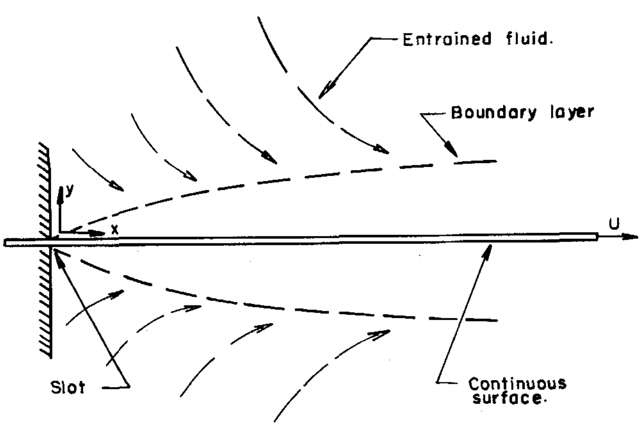

Fig. 1. Boundary layer on a moving continuous flat surface.

Energy

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} \quad (3)$$

Diffusion

$$u\frac{\partial c_A}{\partial x} + v\frac{\partial c_A}{\partial y} = D\frac{\partial^2 c_A}{\partial y^2} \quad (4)$$

The boundary conditions are:

Momentum

$$\begin{aligned} u &= U_f \quad \text{at} \quad y = 0 \\ v &= v_w(x) \quad \text{at} \quad y = 0 \\ u &\to 0 \quad \text{as} \quad y \to \infty \end{aligned} \quad (5)$$

Energy

$$\begin{aligned} T &= T_w \quad \text{at} \quad y = 0 \\ T &\to T_\infty \quad \text{as} \quad y \to \infty \end{aligned} \quad (6)$$

Diffusion

$$\begin{aligned} c_A &= c_{Aw} \quad \text{at} \quad y = 0 \\ c_A &\to c_{A\infty} \quad \text{as} \quad y \to \infty \end{aligned} \quad (7)$$

By introducing the stream function, ψ, defined by

$$u = \frac{\partial \psi}{\partial y} \text{ and } v = -\frac{\partial \psi}{\partial x} \quad (8)$$

which satisfies the equation of continuity, and making the following substitutions (assuming f, θ, ϕ to be functions of η only)

$$f = \frac{\psi}{\sqrt{\nu x U_f}}, \qquad \eta = y\sqrt{\frac{U_f}{\nu x}}$$
$$\theta = \frac{T - T_\infty}{T_w - T_\infty}, \qquad \phi = \frac{c_A - c_{A\infty}}{c_{Aw} - c_{A\infty}} \quad (9)$$

one finds that the boundary layer equations of momentum, energy, and diffusion become:

Momentum

$$f''' + \tfrac{1}{2} ff'' = 0 \quad (10)$$

Energy

$$\theta'' + \tfrac{1}{2} N_{Pr} f\theta' = 0 \quad (11)$$

Diffusion

$$\phi'' + \tfrac{1}{2} N_{Sc} f\phi' = 0 \quad (12)$$

The boundary conditions become:

$$\begin{aligned} f_w &= -2v_w(x)\sqrt{\frac{x}{\nu U_f}} \quad \text{at} \quad \eta = 0 \\ f' &= 1.0 \quad \text{at} \quad \eta = 0 \\ f' &\to 0 \quad \text{as} \quad \eta \to \infty \\ \theta &= 1.0 \quad \text{at} \quad \eta = 0 \\ \theta &\to 0 \quad \text{as} \quad \eta \to \infty \\ \phi &= 1.0 \quad \text{at} \quad \eta = 0 \\ \phi &\to 0 \quad \text{as} \quad \eta \to \infty \end{aligned} \quad (13)$$

The first boundary condition determines the transverse velocity at the moving continuous surface. In obtaining Equation (10), f is assumed to be a function of η only.

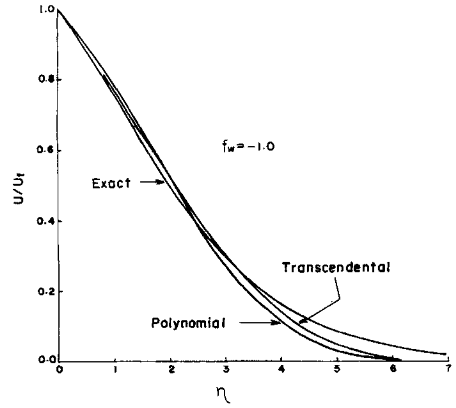

Fig. 2. Comparison of exact velocity profile with the profiles obtained using the polynomial and transcendental approximations, $f_w = -1.0$.

Therefore the value of f at $\eta = 0$ must be a constant. This means $v_w(x)$ must assume the form

$$v_w(x) \sim \frac{1}{\sqrt{x}} \tag{14}$$

if similar solutions are to be obtained. It has been shown by Erickson, et al. (4), that this is indeed the case when the rate of suction or injection is controlled by heat or mass transfer. Equations (10), (11), and (12) have been solved numerically, the momentum equation being tabulated by Erickson and Fan (7), the transfer coefficients appearing in the work by Erickson, et al. (5).

APPROXIMATE SOLUTIONS TO THE MOMENTUM EQUATION

Three different approaches to the solution of the momentum equation are examined. First, the integral method is used to solve the momentum equation by assuming a polynomial profile. Second, the integral momentum equation and the integral mechanical energy equation are used with a profile function of the form suggested by Hanson and Richardson (8), that is,

$$\frac{u}{U_f} = \frac{\exp\{-\exp(a + b\eta)\}}{\exp\{-\exp(a)\}} \tag{15}$$

This profile has never been employed in conjunction with the laminar boundary layer on a continuous moving surface. Third, a new power series is proposed, which uses f' for the independent variable instead of η.

Polynomial Profile

The integral momentum equation for the boundary layer on a moving continuous flat sheet with suction or injection is

$$\frac{d}{dx}\int_0^\delta u^2 dy = -\nu \left.\frac{\partial u}{\partial y}\right|_{y=0} + v_w(x) U_f \tag{16}$$

If the velocity profiles are similar, the dimensionless velocity, u/U_f, can be expressed as a function of the dimensionless distance from the wall, y/δ. The dimensionless velocity is approximated by a fourth order polynomial

$$\frac{u}{U_f} = a_0 + a_1\left(\frac{y}{\delta}\right) + a_2\left(\frac{y}{\delta}\right)^2 + a_3\left(\frac{y}{\delta}\right)^3 + a_4\left(\frac{y}{\delta}\right)^4 \tag{17}$$

which is made to fit the boundary conditions

$$\begin{aligned} u &= U_f \quad \text{at} \quad y = 0 \\ u &= 0 \quad \text{at} \quad y = \delta \\ \frac{\partial u}{\partial y} &= 0 \quad \text{at} \quad y = \delta \\ v_w(x)\left.\frac{\partial u}{\partial y}\right|_{y=0} &= \nu \left.\frac{\partial^2 u}{\partial y^2}\right|_{y=0} \quad \text{at} \quad y = 0 \\ \frac{\partial^2 u}{\partial y^2} &= 0 \quad \text{at} \quad y = \delta \end{aligned} \tag{18}$$

The velocity profile then becomes

$$\frac{u}{U_f} = 1 - \frac{6G}{3G+1}\left(\frac{y}{\delta}\right) - \frac{6}{3G+1}\left(\frac{y}{\delta}\right)^2 + \frac{2(3G+4)}{3G+1}\left(\frac{y}{\delta}\right)^3 - \frac{3(G+1)}{3G+1}\left(\frac{y}{\delta}\right)^4 \tag{19}$$

where $G = 2\nu/v_w(x)\delta(x)$ is assumed to be a constant. When this expression is substituted in Equation (16), one obtains

$$\delta = \sqrt{\frac{4(3G^2 + 3G + 1)}{\alpha G(3G+1)}}\sqrt{\frac{\nu x}{U_f}} \tag{20}$$

where

$$\alpha = \frac{23G^2 + 19G + 4}{14(3G+1)}$$

Since

$$v_w(x) = -\frac{f_w}{2}\sqrt{\frac{\nu U_f}{x}}$$

it can be shown that

$$G = \frac{2\nu}{v_w(x)\delta(x)} = \frac{-4}{f_w\sqrt{\dfrac{4(3G^2+3G+1)}{\alpha G\,(3G+1)}}} \tag{21}$$

is a constant, as was assumed earlier. The value of G depends only upon the injection parameter, f_w. The values for the function, G, are tabulated in Table 1.

The momentum thickness, θ, is defined as

$$\theta = \frac{1}{U_f^2}\int_0^\infty u^2 dy \tag{23}$$

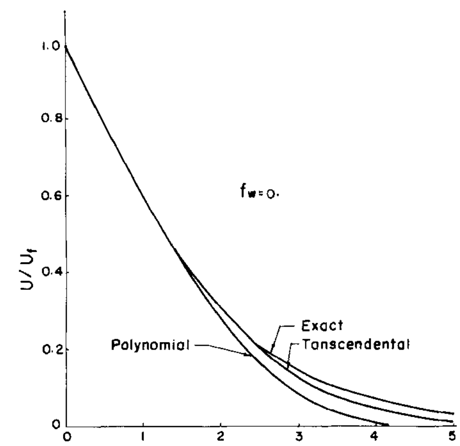

Fig. 3. Comparison of exact velocity profile with the profiles obtained using the polynomial and transcendental approximations, $f_w = 0$.

and is a measure of the loss of momentum due to friction. The displacement thickness, δ^*, is defined as

$$\delta^* = \frac{1}{U_f} \int_0^\infty u dy \tag{24}$$

These two boundary layer parameters can be determined by the substitution of Equation (19) into Equations (23) and (24). The results are presented in Table 2. A comparison of exact profiles and profiles from the integral method using the polynomial profile appears in Figs. 2, 3, and 4.

TABLE 1. THE FUNCTION G FOR USE IN THE POLYNOMIAL PROFILE

f_w	G
−1.0	0.615
−0.5	1.46
−0.1	8.29
0.0	∞
0.1	−8.80
0.5	−1.95
1.0	−1.06

polynomial is used to approximate the velocity profile.

The transcendental form of the profile function has been suggested by Hanson and Richardson (8) and is written as

$$\frac{u}{U_f} = \xi(\eta) = \frac{\exp\{-\exp(a + b\eta)\}}{\exp\{-\exp(a)\}}. \tag{25}$$

The two parameters of the profile function can be evaluated from the momentum and mechanical energy integral equations.

The integral mechanical energy equation for the laminar boundary layer on a moving continuous flat surface with suction or injection can be derived. Since this derivation has not been available in open literature, a brief treatment is given here. Multiplying both sides of the momentum equation, Equation (2), by u, integrating the resulting expression from $y = 0$ to $y = \infty$, and substituting for v from the equation of continuity, one obtains

$$\int_0^\infty \left\{ u^2 \frac{\partial u}{\partial x} + u \left(v_w(x) - \int_0^y \frac{\partial u}{\partial x} dt \right) \frac{\partial u}{\partial x} \right\} dy = \nu \int_0^\infty u \frac{\partial^2 u}{\partial y^2} dy$$

TABLE 2. CHARACTERISTIC BOUNDARY LAYER PARAMETERS FOR MOVING CONTINUOUS FLAT SURFACES WITH SUCTION OR INJECTION

	$\delta/\sqrt{\frac{\nu x}{U_f}}$			$\theta/\sqrt{\frac{\nu x}{U_f}}$			$\delta^*/\sqrt{\frac{\nu x}{U_f}}$		
f_w	EXACT	TRANS.	INTEGRAL	EXACT	TRANS.	INTEGRAL	EXACT	TRANS.	INTEGRAL
−1.0	8.20	5.92	6.50	1.41	1.44	1.40	2.37	2.26	2.18
−0.5	7.30	5.47	5.49	1.12	1.14	1.09	1.97	1.88	1.75
−0.1	6.55	5.09	4.83	0.930	0.941	0.897	1.68	1.61	1.47
0.0	6.37	5.18	4.68	0.888	0.897	0.854	1.62	1.55	1.40
0.1	6.18	4.90	4.55	0.848	0.856	0.814	1.55	1.51	1.35
0.5	5.46	4.50	4.10	0.708	0.713	0.677	1.33	1.28	1.15
1.0	4.66	4.00	3.76	0.573	0.575	0.549	1.09	1.06	0.957

Transcendental Profile

One of the approximations used in developing the integral boundary layer equations for use with the polynomial form of the profile is the finiteness of the boundary layer thickness. This is to say that at a certain finite point, $y = \delta$, the velocity and its first few derivatives are equal to zero. This approximation can be removed from the integral procedure if a transcendental function instead of

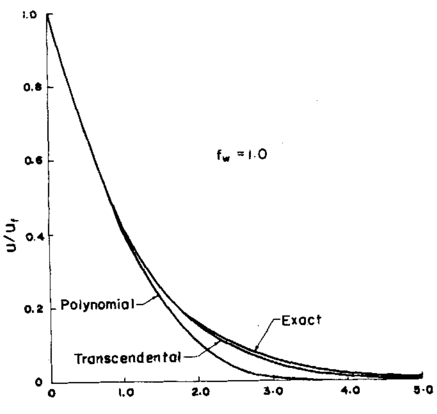

Fig. 4. Comparison of exact velocity profile with the profiles obtained using the polynomial and transcendental approximations, $f_w = 1.0$.

Integrating some of the integrals appearing in this equation by parts gives

$$\int_0^\infty \left\{ u \; v_w(x) \frac{\partial u}{\partial y} \right\} dy = -\frac{v_w(x) \; U_f^2}{2}$$

$$\int_0^\infty \left\{ u \frac{\partial u}{\partial y} \left[\int_0^y \frac{\partial u}{\partial x} dt \right] \right\} dy = -\int_0^\infty \frac{u^2}{2} \left(\frac{\partial u}{\partial x} \right) dy$$

$$\int_0^\infty \left\{ u \frac{\partial u^2}{\partial y^2} \right\} dy = -U_f \left(\frac{\partial u}{\partial y} \right)_w - \int_0^\infty \left(\frac{\partial u}{\partial y} \right)^2 dy$$

Using these relations the integral mechanical energy equation is obtained in the form

$$\int_0^\infty \left\{ \frac{3}{2} u^2 \frac{\partial u}{\partial x} \right\} dy - \frac{v_w(x) \; U_f^2}{2} = -\nu \left[U_f \left(\frac{\partial u}{\partial y} \right)_w + \int_0^\infty \left(\frac{\partial u}{\partial y} \right)^2 dy \right]$$

This can be rewritten as

$$\frac{1}{2} \frac{d}{dx} \int_0^\infty u^3 dy - \frac{v_w(x) \; U_f^2}{2} = -\nu \left(U_f \left(\frac{\partial u}{\partial y} \right)_w + \int_0^\infty \left(\frac{\partial u}{\partial y} \right)^2 dy \right) \tag{26}$$

The integral momentum equation that will be used with the above equation to evaluate the parameters in Equation (25) is identical to Equation (16) except that the upper limit of integration becomes ∞ instead of δ. After the profile function, expression (25), has been substituted into Equations (16) and (26) the following results are obtained. The integral momentum equation becomes

$$b = \frac{f_w + \sqrt{f_w^2 + 8(\exp a)(\exp(2\exp a))E_1(2\exp a)}}{4\exp a} \tag{27}$$

The integral mechanical energy equation becomes

$$b = \frac{\frac{f_w}{4} + \sqrt{\left(\frac{f_w}{4}\right)^2 + \exp(3\exp a)\, E_1(3\exp a)\left\{\frac{\exp a}{2} - \frac{1}{4}\right\}}}{(\exp a) - 0.5} \tag{28}$$

$E_1(x)$ is the exponential integral and is defined by

$$E_1(x) = \int_x^\infty \frac{e^{-t}\,dt}{t} \tag{29}$$

The manipulation necessary to obtain Equations (27), and (28) are found in Appendix 1.* Equating the right hand members of Equations (27) and (28) gives one equation in the single unknown, a. This equation can be solved by a trial and error technique to yield a value for the constant, a. The calculations were performed in such a manner that e^a was determined to three significant figures. Values of a and b were determined for several values of the parameter, f_w, and are reported in Table 3.

TABLE 3. PARAMETERS FOR THE PROFILE FUNCTION, EQUATION (15)

f_w	e^a	a	b
−1.0	0.600	−0.5108	0.3655
−0.5	1.02	0.0198	0.3127
−0.1	1.55	0.4382	0.2712
0.0	1.71	0.5365	0.2623
+0.1	1.90	0.6418	0.2516
+0.5	2.80	1.0296	0.2166
+1.0	4.39	1.4793	0.1794

The expressions for the momentum and displacement thickness can be derived from their defining equations, Equations (23) and (24), once the form of the profile function has been established. For the transcendental profile these equations are

$$\frac{\theta}{\sqrt{\frac{\nu x}{U_f}}} = \frac{\exp(2\exp a)\, E_1(2\exp a)}{b} \tag{30}$$

for the momentum thickness and

$$\frac{\delta^*}{\sqrt{\frac{\nu x}{U_f}}} = \frac{\exp(\exp a)\, E_1(\exp a)}{b} \tag{31}$$

for the displacement thickness. Values of these parameters for this profile are also reported in Table 2. Table 2 shows the accuracy obtainable from the use of the polynomial and transcendental approximations. In particular the use of the transcendental approximation gives errors in the prediction of the momentum thickness never in excess of 2.2% and errors in the prediction of the displacement thickness never in excess of 5.0%.

The profiles obtained using the transcendental approximation for the velocity profiles are compared with the polynomial profile and the exact solution in Figures 2, 3, and 4. A comparison of the profiles shows that the removal of the restriction, $u = 0$, at some finite point and the use of the transcendental function given by Equation (25) resulted in an improved profile. This is particularly true in the case of suction.

Series Solution

The original solution to the boundary layer momentum equation for flow over a finite flat plate was given by Blasius (9). The solution was obtained in the form of a power series expansion about $\eta = 0$ and an asymptotic expansion valid as $\eta \to \infty$, the two solution being joined at a suitable point. If a suitable variable can be found such that a power series converges for all values of the variable, the asymptotic solution and the joining procedure become unnecessary. A suitable bounded variable for the expansion is f'. For notational convenience defined as

$$\xi = \frac{u}{U_f} = f'$$

The transformation of the Blasius function, f, from η space into ξ space can be accomplished by substituting $g(\xi)$ for $f''(\eta)$ and expanding $g(\xi)$ around the point $\xi = 1$.

Expanding $g(\xi)$ in powers of $(\xi - 1)$ gives

$$g(\xi) = g(1) + (\xi - 1)\frac{dg(1)}{d\xi} + \frac{(\xi-1)^2}{2}\frac{d^2g(1)}{d\xi^2} + \ldots \tag{32}$$

where the notation $\frac{dg(1)}{d\xi}$ is used to indicate that the derivatives are evaluated at $\xi = 1$. Note that

$$g(1) = f_w''$$

because of the boundary condition, $\xi = f' = 1$ at $\eta = 0$ (or equivalently at the fluid solid interface). The terms, $\frac{d^n}{d\xi^n}$, can be determined from successive application of

$$\frac{d}{d\xi} = \frac{d\eta}{d\xi}\frac{d}{d\eta} = \frac{d\eta}{df'}\frac{d}{d\eta} = \frac{1}{f''}\frac{d}{d\eta} = \frac{1}{g}\frac{d}{d\eta} \tag{33}$$

For a given amount of suction or injection each of the terms, $\frac{d^ng(1)}{d\xi^n}$, can be written in terms of the single unknown, $g(1)$ (see Appendix 2). $g(1)$ can be evaluated from Equation (32) by the application of the boundary condition, $g(0) = 0$, a statement which makes the velocity gradient and the velocity equal zero at the same place.

For computational purposes the series was computed as

$$g(\xi) = \sum_{n=0}^{s} (\xi - 1)^n C_n \tag{34}$$

* All Appendices are deposited as document 9965 with the American Documentation Institute, Photoduplication Service, Library of Congress, Washington, D. C., and may be obtained for $6.25 for photoprints or $2.50 for 35 mm. microfilm.

The terms necessary to compute C_n for $1 \leq n \leq 7$ are given in Appendix 2, and the values of C_n are given in Table 4.

TABLE 4. THE CONSTANTS C_n FOR USE IN EQUATION (45)

f_w	0.0	0.1	0.5	1.0
C_0	-4.406×10^{-1}	-4.734×10^{-1}	-6.043×10^{-1}	-7.865×10^{-1}
C_1	0.000	-5.000×10^{-2}	-2.500×10^{-1}	-5.000×10^{-1}
C_2	5.674×10^{-1}	5.281×10^{-1}	4.137×10^{-1}	3.179×10^{-1}
C_3	1.891×10^{-1}	1.574×10^{-1}	8.085×10^{-2}	3.860×10^{-2}
C_4	1.218×10^{-1}	8.987×10^{-2}	3.048×10^{-2}	9.142×10^{-3}
C_5	9.743×10^{-2}	6.456×10^{-2}	1.457×10^{-2}	2.752×10^{-3}
C_6	8.943×10^{-2}	5.271×10^{-2}	7.881×10^{-3}	9.366×10^{-4}
C_7	8.813×10^{-2}	4.655×10^{-2}	4.645×10^{-3}	3.496×10^{-4}

This method is exact if the series in Equation (34) converges and if a sufficient number of terms are taken in the series. The results of this method are given only for $f_w \geq 0$ because for negative values of f_w, $C_n < C_{n+1}$ for n sufficiently large.

The dimensionless wall shearing stress is given by

$$\frac{x\,\tau_w}{\mu\,U_f} = -\,g(1)\ \sqrt{N_{Re}} \qquad (35)$$

or equivalently,

$$\frac{x\,\tau_w}{\mu\,U_f} = -\,f_w''\ \sqrt{N_{Re}}$$

Table 5 compares the calculated values of f_w'' for $s = 5$, 6 and 7. Since these terms oscillate with s, it is possible to average the values computed for $s = 6$ and $s = 7$ and obtain a result very close to the result obtained by numerical integration of Equation (10). This average value and the exact value are also presented in Table 5.

The accuracy of this series solution is almost as good as the accuracy of Blasius' original series solution for flow over a finite flat plate (9). The method used by Blasius (9, *10*) could be used to determine the coefficients for the power series of Equation (32). The Blasius series for f' and f'' may be written in terms of the unknown f_w'' as

$$f' = 1 + \eta f_w'' + \frac{\eta^2}{2!2} f_w\,f_w'' + \frac{\eta^3}{3!4}\,[f_w''(f_w^2 - 2)] + \ldots$$

$$f'' = f_w'' + \frac{\eta}{2} f_w\,f_w'' + \frac{\eta^2}{2!4}\,[f_w''(f_w^2 - 2)] + \ldots$$

and the coefficients in the expansion.

$$f'' = \sum_{n=0}^{s} (f' - 1)^n\ C_n$$

may be determined by equating like powers of η.

TABLE 5. f_w'' COMPUTED FROM EQUATION (45).

f_w	$s = 5, f_w''$	$s = 6, f_w''$	$s = 7, f_w''$	Average f_w''	Exact f_w''
0.0	−0.4223	−0.4589	−0.4223	−0.4406	−0.4437
0.1	−0.4602	−0.4835	−0.4634	−0.4734	−0.4738
0.5	−0.6013	−0.6055	−0.6031	−0.6043	−0.6040
1.0	−0.7860	−0.7866	−0.7864	−0.7865	−0.7864

SOLUTIONS OF THE ENERGY AND DIFFUSION EQUATIONS

In the previous section, momentum transfer on a continuous moving flat plate was considered for transverse velocities of the type produced when the rate of heat transfer or mass transfer controls the amount of suction or injection. In this section approximate methods are presented for the solution of the energy and diffusion equations. Again, the only type of suction or injection considered is that which allows similar solutions to the boundary layer equations. The similarity between the boundary layer energy and diffusion equations, Equations (3) and (4), is well known. Adopting the notation

$$\Pi = \begin{cases} \Pi_i = \dfrac{c_A - c_\infty}{c_{Aw} - c_{A\infty}} & \text{for mass transfer} \\ \Pi_T = \dfrac{T - T_\infty}{T_w - T_\infty} & \text{for heat transfer,} \end{cases}$$

$$\lambda = \begin{cases} D & \text{for mass transfer} \\ \alpha & \text{for heat transfer,} \end{cases}$$

and

$$\Lambda = \begin{cases} N_{Sc} & \text{for mass transfer} \\ N_{Pr} & \text{for heat transfer,} \end{cases}$$

the energy and diffusion equations can be written as a single expression

$$u\frac{\partial \Pi}{\partial x} + v\frac{\partial \Pi}{\partial y} = \lambda\frac{\partial^2 \Pi}{\partial y^2}\ (\pi - \text{equation}) \qquad (36)$$

$$\Pi = 1 \quad \text{at} \quad y = 0$$

$$\Pi \to 0 \quad \text{as} \quad y \to \infty$$

or in terms of the similarity variable, η:

$$\Pi'' + \tfrac{1}{2}\,\Lambda\,f\,\Pi' = 0 \quad (\Pi - \text{equation}) \qquad (37)$$

with the boundary conditions

$$f = f_w \quad \text{at} \quad \eta = 0$$

$$\Pi = 1 \quad \text{at} \quad \eta = 0$$

$$\Pi \to 0 \quad \text{as} \quad \eta \to \infty$$

where f is the dimensionless stream function in the momentum equation, Equation (10).

Energy Integral Equation

One of the most popular approximate methods is the Von Karman-Pohlhausen integral method. This method was used to obtain approximate solutions to the boundary layer momentum equation in the previous section. Here its use will be extended to the solution of the π-equation.

Integration of expressions in the equation of continuity gives

$$v = v_w(x) - \int_{t=0}^{t=y} \left(\frac{\partial u}{\partial x}\right) dt \qquad (38)$$

Substituting this result into the π-equation, Equation (36), gives

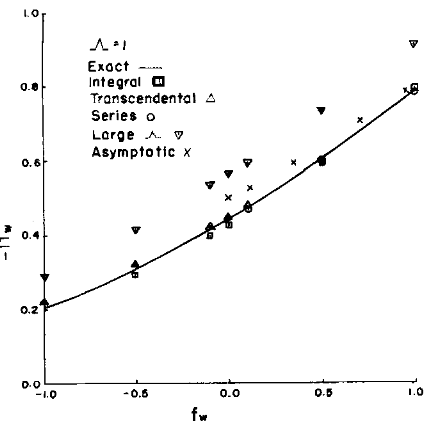

Fig. 5. Comparison of exact heat and mass transfer coefficients with coefficients predicted by approximate methods, A = 1.

$$u\frac{\partial \Pi}{\partial x}+\left[v_w(x)-\int_0^y\left(\frac{\partial u}{\partial x}\right)dt\right]\frac{\partial \Pi}{\partial y}=\lambda\frac{\partial^2\Pi}{\partial y^2} \quad (39)$$

Integration of the terms of the above equation with respect to y over the boundary layer thickness, δ_π, gives

$$\int_0^{\delta_\pi}u\frac{\partial\Pi}{\partial x}dy+\int_0^{\delta_\pi}\left\{\left[v_w(x)-\int_0^y\left(\frac{\partial u}{\partial x}\right)dt\right]\frac{\partial \Pi}{\partial y}\right\}dy=\lambda\int_0^{\delta_\pi}\frac{\partial^2\Pi}{\partial y^2}dy \quad (40)$$

Integration of the double integral by parts, simplification, and use of the boundary conditions, $\Pi = 1$ at $y = 0$, $\Pi = 0$ at $y = \delta_\pi$, and $\partial\Pi/\partial y = 0$ at $y = \delta_\pi$ gives

$$U_f\frac{d}{dx}\int_0^{\delta_\pi}\left(\frac{u}{U_f}\right)\Pi dy-v_w(x)=-\lambda\left.\frac{\partial\Pi}{\partial y}\right|_{y=0} \quad (41)$$

It is now assumed that Π can be expressed in terms of a single variable, η_π, that is

$$\Pi(x,y)=\Pi(\eta_\pi) \quad (42)$$

where

$$\eta_\pi=\frac{y}{\delta_\pi}$$

and that δ_π which is a function of x can be related to the momentum thickness, δ, by

$$\frac{\delta_\pi}{\delta}=\Delta \quad (43)$$

where Δ is a constant.

In order to evaluate the integral π-equation, Equation (41), profile functions for u/U_f and Π must be chosen. A suitable velocity profile is the profile used with the integral momentum equation. This profile which is given by Equation (19) is repeated here; however $\Delta\eta_\pi$ is used instead of y/δ.

$$\frac{u}{U_f}=1-\frac{6G[(\Delta)\eta_\pi]}{3G+1}-\frac{6[(\Delta)\eta_\pi]^2}{3G+1}+\frac{2(3G+4)[(\Delta)\eta_\pi]^3}{3G+1}-\frac{3(G+1)[(\Delta)\eta_\pi]^4}{3G+1} \quad (44)$$

where G is defined by Equation (22) and tabulated in Table 1.

In order to satisfy the Reynolds Analogy, Π should take on the same form as u/U_f, that is

$$\Pi=1-\frac{6G(\eta_\pi)}{3G+1}-\frac{6(\eta_\pi)^2}{3G+1}+\frac{2(3G+4)(\eta_\pi)^3}{3G+1}-\frac{3(G+1)(\eta_\pi)^4}{3G+1} \quad (45)$$

The integral π-equation, Equation (41), becomes

$$\Delta^2\delta\frac{d\delta}{dx}\int_0^1\left(\frac{u}{U_f}\right)\Pi d\eta_\pi-\frac{(\Delta)\delta v_w(x)}{U_f}=\frac{-\lambda}{U_f}\left.\frac{\partial\Pi}{\partial\eta_\pi}\right|_{\eta_\pi=0} \quad (46)$$

Making use of the previously obtained expression for δ in Equation (20), and evaluating the above integral by using the assumed form of u/U_f and Π, Equations (44) and (45) give the following sixth-order polynomial equation for Δ.

$$\frac{-12\,G}{\Lambda(3G+1)\left[\dfrac{\delta}{\sqrt{\dfrac{\nu x}{U_f}}}\right]}-\frac{4\Delta}{G\left[\dfrac{\delta}{\sqrt{\dfrac{\nu x}{U_f}}}\right]^{\Delta^2}}+\left[\frac{9G+4}{10(3G+1)}\right]\Delta^2-\left[\frac{12G^2+6G}{10(3G+1)^2}\right]\Delta^3-\left[\frac{15G+8}{35(3G+1)^2}\right]\Delta^4+\left[\frac{27G^2+51G+20}{140(3G+1)^2}\right]\Delta^5-\left[\frac{7G^2+21G-4}{140(3G+1)^2}\right]\Delta^6=0 \quad (47)$$

The above equation was solved for Δ values of f_w from -0.5 to 1 and for $\Lambda = 1$, 10, 100 using Lin's method (*11*). The transfer coefficients can then be computed from the equation

$$\left.\frac{\partial\Pi}{\partial\eta}\right|_{\eta=0}=\frac{\partial\eta_\pi}{\partial y}\frac{\partial y}{\partial\eta}\frac{\partial\Pi}{\partial\eta_\pi}=-\left(\frac{1}{\Delta\delta}\right)\left(\sqrt{\frac{\nu x}{U_f}}\right)\left(\frac{6G}{3G+1}\right) \quad (48)$$

Values of $\frac{1}{\delta}\sqrt{\frac{\nu x}{U_f}}$ corresponding to particular amounts

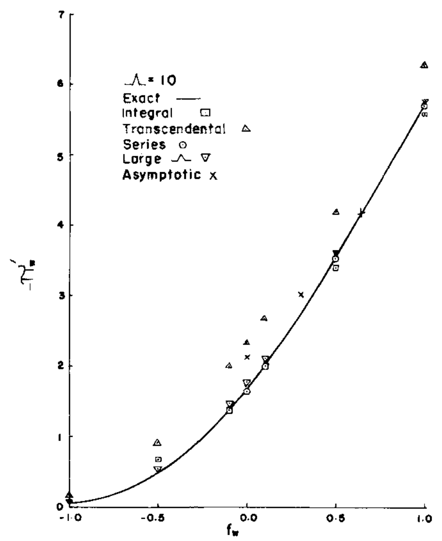

Fig. 6. Comparison of exact heat and mass transfer coefficients with coefficients predicted by approximate methods, A = 10.

of suction or injection can be found in the previous section in Table 2.

The transfer coefficients computed from Equation (48) are compared with the exact numerical results in Figures 5, 6, and 7 and are tabulated in Appendix 3. It can be concluded that this method gives rise to acceptable results, particularly for the case of suction.

No real positive roots were found for $f_w = -1$, $\Lambda = 1$, 10, 100 and for $f_w = -0.5$, $\Lambda = 100$ using Lin's method, Newton's method, or the method of exhaustive search. Since the purpose of using the integral method is to obtain approximate results rapidly, no further techniques were attempted to solve for Δ. It should be noted that the value for the transfer coefficient for $f_w = -1$, $\Lambda = 1$ can be obtained by the Reynolds analogy from the solution of the integral momentum equation.

Transcendental Method

The transfer coefficients, Π_w', for heat and mass transfer can be obtained from the Π-equation, Equation (37), by numerical integration if the function, $f''(\eta)$, is known. That this is a fact can be shown in the following manner. Integrating the Π-equation, Equation (37), once yields

$$\Pi'(\eta) = \Pi_w' \exp\left\{-\frac{\Lambda}{2}\int_{z=0}^{z=\eta} f(z)\,dz\right\} \tag{49}$$

The second integration yields

$$\Pi(\eta) = 1 + \Pi_w' \int_0^\eta \left[\exp\left\{-\frac{\Lambda}{2}\int_0^t f(z)\,dz\right\}\right] dt \tag{50}$$

By using the momentum equation [see Equation (10)]

$$f = -2f'''/f''$$

the inside integral of Equation (50) can be written as

$$\int_0^t f(z)\,dz = -2\int_0^t \frac{df''}{dz}\left(\frac{1}{f''}\right)dz = -2\ln\left\{\frac{f''(t)}{f''_w}\right\} \tag{51}$$

and

$$\exp\left\{-\frac{\Lambda}{2}\int_0^t f\,dz\right\} = \left[\frac{f''(t)}{f''_w}\right]^\Lambda \tag{52}$$

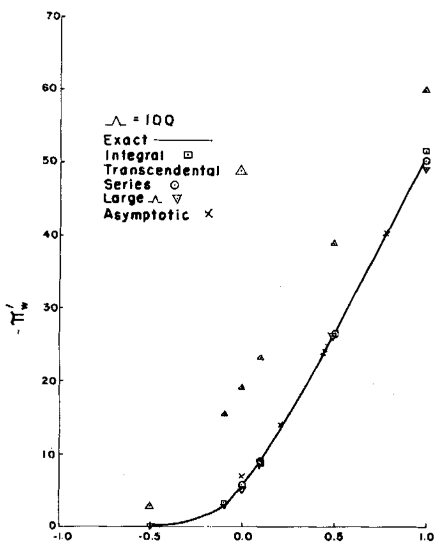

Fig. 7. Comparison of exact heat and mass transfer coefficients with coefficients predicted by approximate methods, $\Lambda = 100$.

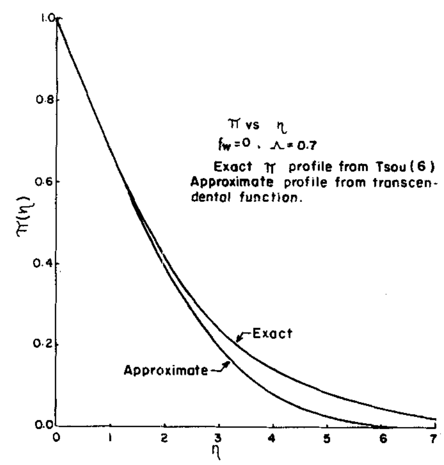

Fig. 8. Comparison of exact Π profile with approximate profile obtained from transcendental approximation.

Equation (50) can now be written as

$$\Pi(\eta) = 1 + \frac{\Pi'_w}{[-f''_w]^\Lambda}\int_0^\eta [-f''(t)]^\Lambda\,dt \tag{53}$$

Application of the boundary condition, $\Pi \to 0$ as $\eta \to \infty$, gives

$$\Pi'_w = \frac{-[-f''_w]^\Lambda}{\int_0^\infty [-f''(\eta)]^\Lambda\,d\eta} \tag{54}$$

The transfer coefficient, Π'_w, can be evaluated by carrying out the indicated integration once a suitable form for $f''(\eta)$ has been chosen. One function that may be used to approximate $f'(\eta)$ is the exponential function of Equation (25):

$$f'(\eta) = \xi(\eta) = \frac{\exp(-\exp(a + b\eta))}{\exp(-\exp a)}$$

Inserting this expression into Equations (54) and (53) yields

$$\Pi'_w = \frac{-b\,\{\exp a\,(\exp(-\exp a))\Lambda\}^\Lambda}{\Gamma(\Lambda, \Lambda \exp a)} \tag{55}$$

and

$$\Pi(\eta) = \frac{\Gamma(\Lambda, \Lambda\exp(a + b\eta))}{\Gamma(\Lambda, \Lambda \exp a)} \tag{56}$$

respectively. Thus the temperature and concentration profiles and transfer coefficients can be expressed in terms of incomplete gamma functions. The intermediate steps used in the derivation of Equation (55) and (56) are presented in Appendix 4.

The transfer coefficients calculated from Equation (55) are plotted in Figures 5, 6 and 7 and tabulated in Appendix 3. The profile given by the values of the parameters a and b can be found in Table 3. Equation (56) is plotted for the case of $\Lambda = 0.7$ and compared with the results of Tsou (6) in Figure 8.

Series Approximation

If the series approximation for the velocity profile, Equation (34), is convergent, it is an approximation only because the exact infinite series is truncated for computational purposes. Hence, it can be expected that the use of this series to determine the transfer coefficient, Π_w', will yield values which are very close to the exact numerical results. If we define ξ by the relation

$$\xi = f' \tag{57}$$

and change the variable of integration in the equation for the transfer coefficient, Equation (54), from η to ξ, we obtain

$$\Pi_w' = \frac{-[-f''_w]^{\Lambda}}{\int_1^0 [-f''(\xi)]^{\Lambda-1}\, d\xi} \tag{58}$$

Numerical values of Π_w were obtained by substituting Expression (34) into the above equation and then using the trapezoidal rule to evaluate the integral. The values of the transfer coefficients so obtained are also plotted in Figures 5, 6 and 7 and tabulated in Appendix 3.

Transfer Coefficients for Large Values of Λ(N_{PR} or N_{Sc})

When Λ is large, the concentration or thermal boundary layer is much thinner than the momentum boundary layer, that is

$$\delta_\pi << \delta \tag{59}$$

and all the heat or mass transfer takes place near the moving continuous surface. For this situation an approximation to the Π-equation, Equation (37), can be obtained by linearization and Π'_w can be obtained in closed form.

The function, f, which appears in the Π-equation, can be linearized by expanding it in a power series about the point $\eta = 0$ as follows:

$$f = f_w + \eta f'_w + \frac{\eta^2}{2!} f''_w + \ldots \tag{60}$$

If the heat or mass transfer takes place near the moving surface as assumed, a good approximation can be obtained by dropping terms of order η^2 and higher. Thus

$$f = f_w + \eta f'_w = f_w + \eta \tag{61}$$

The Π-equation, Equation (37), becomes

$$\Pi'' + \tfrac{1}{2}\Lambda(f_w + \eta)\Pi' = 0 \tag{62}$$

$$\Pi = 1 \quad \text{at} \quad \eta = 0$$

$$\Pi \to 0 \quad \text{as} \quad \eta \to \infty$$

The transfer coefficient can be determined from the above equation and is

$$\Pi'_w = \frac{-\left(\frac{\Lambda}{\pi}\right)^{1/2}}{\exp\left(\frac{\Lambda f_w^2}{4}\right) \operatorname{erfc}\left(\frac{\sqrt{\Lambda} f_w}{2}\right)} \tag{63}$$

The values for Π'_w obtained from Equation (63), are also plotted in Figures 5, 6, and 7 and tabulated in Appendix 3. The values obtained are accurate to within about 5% for

$$|f_w| \leqq 0.5, \quad \Lambda \geqq 10$$

Transfer Coefficients for Large Values of Suction

An asymptotic expression (11) which is valid for large values of suction has been presented in a previous paper (4). For completeness the result is presented here.

$$\pi'_w = -\Lambda \sqrt{\frac{1}{2(1-n)(1+\Lambda)}} \tag{64}$$

For mass transfer n is defined by

$$n = \frac{c_{A\infty} - c_{Aw}}{\rho - c_{Aw}}$$

For heat transfer n is defined by

$$n = \frac{(T_w - T_\infty)}{\lambda} C_p \tag{65}$$

The coefficients obtained from Equation (65) are also plotted in Figures 5, 6 and 7 and tabulated in Appendix 3.

THE COOLING OF A MOVING CONTINUOUS FLAT SHEET

The problem of determining the temperature of a moving continuous flat sheet of half width B as it passes through a tank of fluid has been investigated by Erickson, Cha, and Fan (5) for the case where the temperature of fluid far from the plate is maintained at a constant temperature. Erickson, Cha and Fan (5) solved the problem by an integral method and by the method of finite differences for $N_{Re} = 625\,x$, $\rho_s c_s B/\rho c_p = 0.01$, and several values of the Prandtl number. The solution which was obtained by the method of finite differences will be called the exact solution.

The equation of energy for the fluid can be written in terms of the dimensionless temperature θ, defined by

$$\theta = \frac{T - T_\infty}{T_i - T_\infty} \tag{66}$$

where T_i is the temperature of the extruded sheet at the point where it issues from the slot. It is

$$u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} = \alpha \frac{\partial^2 \theta}{\partial y^2} \tag{67}$$

If the temperature across the cross section of the extruded sheet can be assumed constant, a valid assumption if the thickness of the sheet 2B is small, the energy equation for the sheet is

$$-B\; k_s \frac{d^2\theta_s}{dx^2} + B\rho_s\; c_s\; U_s \frac{d\theta_s}{dx} = k\frac{\partial \theta}{\partial y}\bigg|_{y=0} \tag{68}$$

The subscript s refers to the sheet.

If it is further assumed that conduction in the axial direction is negligible, a good assumption if k_s is small or U_s is large or both, Equation (68) can be written as

$$B\rho_s\; c_s\; U_s \frac{d\theta_s}{dx} = k\frac{\partial \theta}{\partial y}\bigg|_{y=0} \tag{69}$$

For large values of N_{Pr} the thermal boundary layer will be much thinner than the momentum boundary layer. Thus, the temperature distribution for large N_{Pr} falls to zero much closer to the moving surface than does the velocity. Griffith (3) used these facts to justify neglecting the variation in velocity in the region where temperature changes. This is equivalent to setting $u = U_s$ and $v = 0$ in Equation (67). Griffith presented a solution to the linearized equation for the case of a moving continuous cylinder. No exact numerical solutions or experimental results were reported.

It will be shown here that Griffith's approximation is valid for the problem of the moving continuous flat sheet. The order of the approximation $u = U_s$, $v = 0$ can be shown by recalling the definitions for u and v in terms of the Blasius function, f, and expanding f in a Taylor series. This gives

$$\frac{u}{U_s} = f' = 1 + \eta f''_w + \ldots \tag{70}$$

and

$$v = \frac{1}{2}\sqrt{\frac{\nu U_s}{x}}\,(\eta f' - f)$$

$$= \frac{1}{2}\sqrt{\frac{\nu U_s}{x}}\left(\eta + \eta^2 f''_w - \eta - \frac{\eta^2}{2!} f''_w + \ldots\right) \quad (71)$$

Neglecting all terms of order η or higher gives

$$u = U_s, \quad v = 0 \quad (72)$$

Physically, this means that there is no relative motion between the sheet and the fluid. That is, the fluid flow is one dimensional and is equal to the velocity of the sheet everywhere. Substituting the expressions from Equation (72) into Equation (67) gives the linearized energy equation for the fluid as

$$U_s \frac{\partial \theta}{\partial x} = \alpha \frac{\partial^2 \theta}{\partial y^2} \quad (73)$$

Division of both sides of Equation (69) by $\rho c U_s$ gives

$$\beta \frac{d\theta_s}{dx} = \left(\frac{1}{N_{Pr}}\right)\left(\frac{1}{N_{Re}}\right) \frac{\partial \theta}{\partial y}\bigg|_{y=0} \quad (74)$$

where

$$\beta = \frac{\rho_s \, c_s \, B}{\rho c}$$

Equation (73) can be rearranged to give

$$\frac{\partial^2 \theta}{\partial y^2} - \left(\frac{N_{Re}}{x}\right)(N_{Pr}) \frac{\partial \theta}{\partial x} = 0 \quad (75)$$

Equations (74) and (75) are subject to the boundary conditions

a. $\theta_s = 1$ at $x = 0$, all y
b. $\theta = 0$ at $x = 0$, $y \geq 0^+$
c. $\theta_s = \theta$ at $y = 0$ all x
d. $\theta \to 0$ as $y \to \infty$, all x (76)

The solution to this set of linear partial differential equations can be obtained by the use of Laplace transforms. The expression for the temperature of the plate is

$$\theta_s = \exp\left\{ \left(\frac{1}{\beta}\right)^2 \left(\frac{x}{N_{Re}}\right) \left(\frac{1}{N_{Pr}}\right) x \right\} \operatorname{erfc}\left\{ \frac{\sqrt{x}}{\beta \sqrt{\left(\frac{N_{Re}}{x}\right) N_{Pr}}} \right\} \quad (77)$$

The plate temperature is plotted as a function of x for $\frac{N_{Rex}}{x} = 625$, $\beta = 0.01$, and $N_{Pr} = 1, 10, 1{,}000$ in Figure 9. The results obtained by Erickson, Cha, and Fan (5) are also plotted in Figure 9 for comparison.

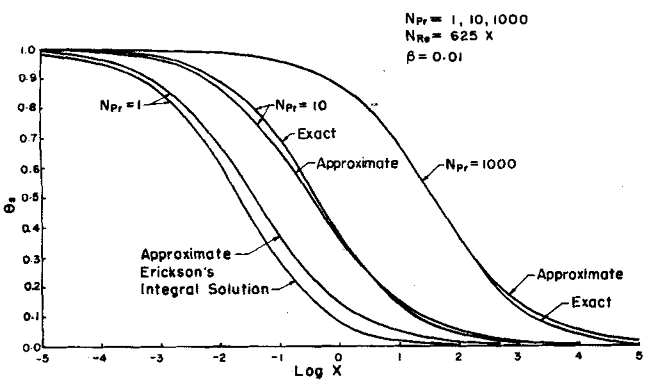

Fig. 9. Surface temperature as a function of distance on a moving continuous flat sheet.

DISCUSSION AND CONCLUSION

Solutions to the momentum equation, Equation (2), and the π-equation, Equation (36), were obtained using various techniques, most of which permit hand calculation of the properties of interest. For the momentum boundary layer the new integral method which uses a transcendental velocity profile gives results that are somewhat superior to those obtained using the polynomial profile. For the values of the injection parameter considered, the transcendental method gives values for the momentum thickness that deviate less than 2.2% from the exact values even though the selected profile does not satisfy the compatibility condition,

$$f_w''' = -\frac{1}{2} f_w \, f_w''$$

The use of a velocity profile of the form

$$\frac{u}{U_f} = \xi(\eta) = \frac{\exp\{-\exp(\alpha + \beta\eta + \gamma\eta^2)\}}{\exp(-\exp \alpha)}$$

was also attempted, where α, β, γ were evaluated from the mechanical energy and momentum integral equations, and the wall compatibility condition. However, the resulting equations are too complex for the hand calculation of the constants α, β, and γ.

When the integral method presented by Sakiadis (*1*) is modified to account for suction and injection it gives acceptable results for the momentum boundary layer parameters. For most values of the injection parameter, the results are not as good as those obtained using the transcendental method; however, they are somewhat easier to obtain using hand computation.

The series solution which is proposed can produce an exact solution if a sufficient number of terms are taken and if the series converges. Furthermore, the one unknown, f''_w, can be evaluated by solving a single polynomial equation. These advantages are balanced by the lack of convergence for the case of injection and by the complexity of the terms, $\frac{d^n f''}{d\xi^n}$.

The integral method provided acceptable solutions to the π-equation, Equation (36), over the range $-.5 \leq f_w \leq 1$ for all values of Λ considered when solutions were obtainable. For the case $f_w = -.5$, $\Lambda = 100$ and $f_w = -1$, $\Lambda = 1, 10, 100$, no values were found for Δ.

The solutions of the Π equation based on the transcendental function are in good agreement with the exact values for $\Lambda = 1$. However, there is poor agreement for $\Lambda = 10$, and 100. This is because the identity, $f = 2f'''/f''$, was used in deriving Equation (53). The transcendental profile does not satisfy this condition in the important region near the moving continuous surface. For large values of Λ in this region is critical and the transcendental profile should not be used.

The solution proposed for large Λ is seen to be valid for $\Lambda = 10$ and 100. The series solution also gives excellent results but these are obtained at the expense of numerical labor. The integral which appears in the equation for the transfer coefficient must be evaluated numerically.

The solution for large values of the suction parameter is seen to yield improved results as Λ becomes large for the same value of f_w. This is true because the solution is asymptotic as $n \to 1$. Equation (13) can be written in terms of n as

$$f_w = \frac{2n\Pi'_w}{\Lambda} \qquad (78)$$

Combining the expressions given in Equations (78) and (64) yields

$$\frac{(1 + N_{Sc})\, f_w}{2} = \frac{n^2}{1 - n} \qquad (79)$$

For a given positive value of f_w, n will approach its asymptotic value of 1 as N_{Sc} becomes large.

The approximation suggested by Griffith (3) can be used to predict the surface temperature of a moving continuous flat sheet which is being cooled as it passes through a tank of fluid. As shown in Figure 9 the approximation is very good when the Prandtl number is large.

SUMMARY

In this work the momentum and π-boundary layers on a moving continuous flat sheet are investigated by several new and modified methods. The methods which are most generally valid are the transcendental method for the momentum equation and the integral method for the energy equation. For the special case of large Λ and the special case of high suction values, closed form solutions are obtainable that predict the transfer coefficient with considerable accuracy. Some aspects of the behavior of the boundary layer on the continuous moving surface, which have not been known, are also discussed in this paper.

NOTATION

A_1 = $\left(\frac{1}{\beta}\right)\left(\frac{x}{N_{Rex}}\right)\left(\frac{1}{N_{Pr}}\right)$ dimensionless
A_2 = $\left(\frac{N_{Rex}}{x}\right)(N_{Pr})$, L^{-1}
a = constant in transcendental function, dimensionless
$a_0, a_1, \ldots$ = coefficients used in polynomial profile, dimensionless
B = half width of flat surface, L
b = constant in transcendental function, dimensionless
C_n = coefficients used in the expansion of g, dimensionless
c = heat capacity of fluid, L^2/t^2T
c_s = heat capacity of moving sheet, L^2/t^2T
c_A = mass concentration of species, A, M/L^3
c_{Aw} = mass concentration of species A at the fluid-solid interface, M/L^3
D = diffusivity, L^2/t
F = function of z used in asymptotic approximation, dimensionless
f = stream function in terms of η, dimensionless
G = constant used in polynomial profile, dimensionless
g = velocity gradient in terms of ξ, dimensionless
h = distance larger than the thermal boundary layer thickness, L
I_1, I_2, I_3 = integrals defined by Equations (29a), (31a), and (32a)
K = $-\left(\frac{\partial \phi}{\partial \eta}\right)_w$, dimensionless
N_{Pr} = $c\mu/k$ = Prandtl number, dimensionless
N_{Re} = $U_f x/\nu$ = Reynolds number, dimensionless
N_{Sc} = ν/D = Schmidt number, dimensionless
n = $\frac{C_{A\infty} - C_{Aw}}{\rho - C_{Aw}}$, dimensionless
p = Laplace transform variable
T = temperature, T
T_w = temperature at the fluid-solid interface, T
T = temperature outside the thermal boundary layer, T
t = variable of integration, dimensionless
U_f = velocity of the continuous surface, L/t
U_s = velocity of the continuous surface, L/t
u = fluid velocity component in x direction, L/t
v = fluid velocity component in y direction, L/t
v_w = fluid velocity component in the y direction at the fluid, solid interface, L/t
x = coordinate of Cartesian system of axes, L
y = coordinate of Cartesian system of axes, L
y_1 = variable of integration
z = $K\eta$, dimensionless

Greek Letters

α = $k/\rho c$ = thermal diffusivity, L^2/t
β = $\frac{\rho_s\, c_s\, B}{\rho c}$, L
$\Gamma(A, B)$ = incomplete gamma function
Δ = ratio of π boundary layer thickness to momentum boundary layer thickness, dimensionless
δ = momentum boundary layer thickness, L
δ_π = boundary layer thickness, L
δ^* = displacement thickness, L
η = $y\sqrt{U_f/\nu x}$, dimensionless variable
η_π = y/δ_π, dimensionless variable
θ = $(T - T_\infty)/(T_w - T_\infty)$, dimensionless temperature
θ = momentum thickness, L
Λ = N_{Sc} or N_{Pr}, dimensionless
λ = D or α, L^2/t
μ = viscosity, M/Lt
ν = kinematic viscosity, L^2/t
ξ = f', dimensionless
Π = concentration or temperature boundary layer equation written in terms of η
π = concentration or temperature boundary layer equations written in terms of x and y
ρ = density, M/L^3
τ_w = shear stress on continuous flat surface, M/Lt^2
ϕ = $\frac{c_A - c_{A\infty}}{c_{Aw} - c_{A\infty}}$, dimensionless concentration
ψ = stream function, L^2/t

Subscripts

A, B = species in binary system
s = solid
w = value taken at fluid solid interface

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